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Deformation of a Rectangular Guide

by

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Abstract

The authors develop the following Taylor's expansion about $R = \infty$,

$$\frac{\pi}{2}\Big[J_\nu(ka)\, N_\nu(ka + kR) - J_\nu(ka + kR)\, N_\nu(ka) \Big]_{\nu = cR}$$

$$= \frac{a}{R}\,\frac{\sin x}{x} + \frac{1}{2}\,\frac{a^2}{R^2}\left\{\frac{\sin x}{\gamma x} - \left(1 + \frac{1}{\gamma}\right)\cos x\right\} + \ldots \quad .$$

This is applied to a study of the modification in cut-off wave number k associated with the TM mode of a rectangular guide when the guide is "buckled" so that the rectangular cross-section takes the shape of a wedge formed by two arcs of concentric circles (one of radius R, the other of radius R + a) and two radial segments. It should be emphasized that the order of the Bessel functions is proportional to R, $\nu = cR$ and c constant depending on a dimension of the guide and on the mode. A similar formula and application are worked out for the TE modes.

Contents

1. Introduction

The effect of deformations of the shape of a guide on the field configurations, the cut-off frequencies, phase velocities, and losses in guides is important for variations in these quantities can seriously affect operation. Such distortions are, unfortunately, unavoidable because no guide is manufactured true to shape and because handling and use cause deformations. While variations in the fundamental guide quantities under deformation is to be expected the practical question is, how serious are the variations? Or, how stable are the quantities?

This paper studies the effect of one type of deformation on all the modes in a rectangular guide. The deformation in question arises as follows. We consider a figure formed by two infinitely long coaxial cylinders and by two radial half-planes meeting along the axis of the cylinders. The structure thus obtained is a sector of a coaxial guide, or briefly, a coaxial sector. A cross-section taken perpendicular to the axis is shown in Fig. 1. If the dimensions a and b are kept constant while the radius R of the inner cylinder is permitted to increase towards infinity, the coaxial sector tends to a rectangular guide with dimensions a and b. Hence for large R the coaxial sector is a deformation of the rectangular guide, the deformation being in cross-sectional shape as opposed to deformation along the direction of the axis.

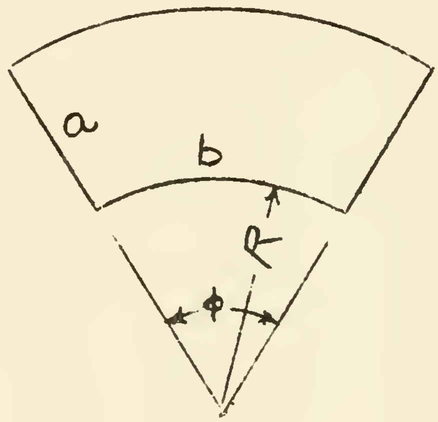

FIG 1

If one calculates a guide quantity such as phase velocity, wave length, etc., for a coaxial sector for large R and compares this to the corresponding quantity for a rectangle, one may attribute any difference in this quantity to a deformation of the rectangle.

It is possible to obtain the respective quantities for the true rectangle as a limiting case of the theory of the coaxial sector. Moreover by comparing the value of any one quantity such as cut-off frequency for the rectangle and for the coaxial sector one obtains the effect of this deformation on the rectangle.

The basic accomplishment of this paper is the development of the mathematics of the coaxial sector for the case in which the angle φ of Figure 1 is permitted to vary while R varies so as to hold b constant. In this development quantities

belonging to the coaxial sector are computed to the second order in $\frac{1}{R}$ for large R. The method permits the calculation of higher order terms but only with excessive labor.

Specifically, the mathematical problem is as follows. The coaxial sector is best approached by the use of cylindrical coordinates. In this coordinate system one is led readily, as will be seen in Article 2 below, to the problem of finding the values of k satisfying the equations

$$J_\nu(kR)\, N_\nu(kR + ka) - J_\nu(kR + ka)\, N_\nu(kR) = 0 \qquad (1)$$

for T M modes and

$$J'_\nu\,(kR)\, N'_\nu\,(kR + ka) - J'_\nu\,(kR + ka)\, N'_\nu\,(kR) = 0 \qquad (2)$$

for T E modes.

In these equations, J_ν and N_ν are the Bessel functions of first and second kind, of order ν; where ν is a real number depending upon the angle φ (Fig. 1). The factor k in the arguments of these functions has the physical significance that it is inversely proportional to the free-space cut-off wave length. The solution of equations such as (1) and (2) for values of k is difficult even when ν is integral. It is all the more difficult when ν is any real number. There is a further difficulty connected with the determination of the k values in this problem, namely that if the sector is to approach a particular rectangular guide certain dimensions of the sector must be kept constant and, as will be shown below, ν through its dependence on φ, becomes a function of the radius R. Hence any attempt to study the variation in k values with the radius R must take into account that ν, too, is a function of R.

The method employed in this paper is to expand some of the Bessel functions involved in equations (1) and (2) by Taylor's theorem. Then by rearrangement of the terms of the resulting series and the use of elementary relations between independent solutions of the Bessel equation, to obtain all terms of first and second order in the variable $\frac{1}{R}$. This approach is an alternative to the use of those special asymptotic expressions for the Bessel functions wherein the order is a function of the argument. It did not seem possible to use these very complicated asymptotic expansions for the purposes of this paper.

The final equation obtained for k gives k in implicit form to terms of second order in $\frac{1}{R}$. From this implicit representation k is obtained explicity to two terms of a Taylor's expansion. This may be seen in equations (17) and (23) for the T M and T E modes respectively. The second term represents the major part of

variation in k arising from the deformation.

The major result may be stated thus: If the side a is taken as a unit of length then the variation in k divided by the value of k for the true rectangle is less than one-half of the curvature of the other dimension. For small curvature the same statement applies to the free space cut-off wave length which is $\frac{2\pi}{k}$. As to the $TE_{0,1}$ mode used widely in practice the theory states that curvature of the non-critical dimension (the sides parallel to the direction of the electric field) does not affect the wave number or the cut-off wave length to first order terms; however curvature of the critical dimension does affect the value of k and the cut-off frequency in accordance with the general result stated just above. These predictions of the theory appear to be harmonious with physical intuition.

It should be pointed out that emphasis was laid on k throughout this paper because from a knowledge of k all the desired quantities such as cut-off frequency, wave length in the guide, phase velocity, characteristic wave impedance, field equations, etc., can be obtained. The relationships of these other guide quantities to k are conveniently summarized in Ramo and Whinnery.[1]

A fair amount of work has already been done on the effect of deformations on guide constants. A very limited attack on the problem for rectangular guides and one employing the same deformation as does this paper is made by Schelkunoff.[2] He treats the lowest mode only and by physical arguments and some numerical computation obtains approximate expressions for the field and an approximate evaluation of the effect of the deformation on the cut-off frequency of the lowest mode. The mathematical approach of this paper supplants and extends the treatment in Schelkunoff.

The work of this paper should be compared with that of Buchholz[3] who approaches the problem of the effect of curvature of the axis on a rectangular

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1. Ramo, S. & Whinnery, John R.: Fields and Waves in Modern Radio, John Wiley and Sons, Inc., N. Y., 1944, p. 322

2. Schelkunoff, S. A.: Electromagnetic Waves, D. Van Nostrand Co., Inc., N. Y., pp 328 to 330 and pp 435-437.

3. Buchholz, H.: Der Einfluss der Krümmung von rechteckigen Hohlleitern auf das Phasenmass ultrakurzer Wellen Elektrische Nachrichten - Technik, v. 16, 1939, pp 73-85.

guide by considering the guide to be a section of a torus with a rectangular cross-section, with propagation taking place along the direction of increasing φ. In this form of the problem Buchholz has to solve our equations (1) and (2) for fixed argument but for unknown ν, the order of the Bessel function. His results are valid only for large R and the ν values obtained are approximate.

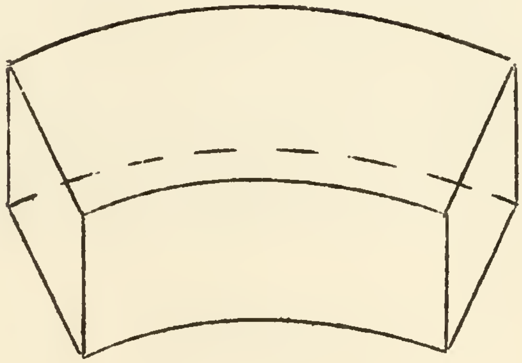

FIG. 2

Study of the effect of deformation of the axis of a rectangular guide has also been made by Jouguet[4,5,6]. The same author and others have considered the same problem in connection with circular guides[6,7,8]. In particular the effect of deformation on the circular guide has been treated by studying special shaped guides such as the elliptic and parabolic which may be regarded as deformations of a circular guide[9,10]. Quite a number of papers, of which one by Bernier[11] is typical, study the effect of arbitrary but small variations in shape of a guide by the method of perturbations.

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4. Jouguet, M.: The Effect of the Curvature of a Wave Guide on Propagation, Compt. Rend. Acad. Sci. (Paris), v. 223, 1946, pp 380-381.

5. Jouguet, M.: The Effect of a Curvature Discontinuity on Propagation in Wave Guides, Compt. Rend. Acad. Sci. (Paris), v. 223, 1946, pp 474-75.

6. Jouguet, M.: On the Propagation of Waves in Curved Guides, Compt. Rend. Acad. Sci. (Paris) v. 222, 1946, pp 537-38.

7. Jouguet, M.: On the Propagation of Electromagnetic Waves in Curved Tubes, Compt. Tend. Acad. Sci. (Paris) v. 222, 1946, pp 440-441.

8. Schelkunoff, S.A.: A Note on Guided Waves in Slightly Non-circular Tubes, Journal of Applied Physics, v. 9, 1938, pp 484-88.

9. Chu, L. J.: Electromagnetic Waves in Elliptic Hollow Pipes of Metal, Journal of Applied Physics, v. 9, 1938, pp 583-591.

10. Spence, R.D. and Wells, C.P.: Propagation of Electromagnetic Waves in Parabolic Pipes, Phys. Rev., v. 62, 1942, pp 58-62.

11. Bernier, J.: Sur les Cavités Electromagnetiques L'Onde Électrique, v. 26, Aug-Sept. 1946, pp 305-317.

2. Basic Theory

Our approach to the deformed rectangular guide is made, as already remarked, by considering the guide formed by the portion of two coaxial circular cylinders which is cut out by two radial half-planes meeting along the common axis (Figure 1).

The basic theory for the field inside the structure of Fig. 1 is known. We refer to the discussion to be found on pp.325-326 and pp.333-335 of Ramo and Whinnery.* The general expression for the axial electric component, E_z, of the TM modes and the axial magnetic component, H_z, of the TE modes can be written as:

$$\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = \left[AJ_\nu(kr) + BN_\nu(kr) \right] \begin{Bmatrix} \sin \nu\varphi \\ \cos \nu\varphi \end{Bmatrix} e^{(\omega t + hz)} , \begin{Bmatrix} \text{TM modes} \\ \text{TE modes} \end{Bmatrix} ,$$

where r, φ, z are cylindrical coordinates;

A and B are constants whose ratio is determined by the boundary conditions on the arc-walls of the guide;

J_ν and N_ν are Bessel functions of the first and second kind, respectively, of order ν;

$\nu = \dfrac{m\pi}{\varphi_0}$, m = 1,2,3,..., with φ_0 the angle of the sector (Fig. 1). This value of ν results from the boundary conditions on the straight walls of the guide, namely $E_z = 0$ for TM modes, $\dfrac{\partial H_z}{\partial \varphi} = 0$ for TE modes, at $\varphi = 0$ and $\varphi = \varphi_0$. It is clear that ν is not, in general, an integer.

ω is the angular frequency of the wave; and h is the propagation constant.

$k^2 = h^2 + \omega^2 \mu_1 \varepsilon_1$, where μ_1 and ε_1 are the permeability and dielectric constant of the dielectric in the guide. This last relationship shows that the value of k determines the value of h, the propagation constant.

By virtue of the boundary conditions on the circular walls of the guide, at r = R and r = R + a, the parameter k is one of a monotonic increasing sequence of "characteristic numbers". These characteristic numbers k are determined as solutions of the transcendental equation (see p.333 of Ramo and Whinnery, loc.cit.)

TM modes: $J_\nu(kR)\, N_\nu(kR + ka) - J_\nu(kR + ka)\, N_\nu(kR) = 0$ (1)

TE modes: $J'_\nu(kR)\, N'_\nu(kR + ka) - J'_\nu(kR + ka)\, N'_\nu(kR) = 0$. (2)

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* loc.cit. We shall use k for the k_c of that text.

in which the parameters R, $\underline{a}$ and ν are to be regarded as constants, as indeed they are for a fixed guide.

We are interested here in the guide of Fig. 1 as a deformation of a rectangle of sides $\underline{a}$ and $\underline{b}$ and in particular in the variation of the numbers $\underline{k}$ when the rectangle is deformed. We can bring about a transition from the deformed guide to the rectangular one by allowing R to approach infinity, keeping $\underline{a}$ fixed and at the same time requiring φ_0 to approach zero in such a way that

$$R\,\varphi_0 = b, \text{ constant}. \tag{3}$$

Then, of course the constants ν must also vary and $\nu = \dfrac{m\pi}{\varphi_0}$ leads to

$$\nu = \frac{m\pi R}{b}, \; m = 1, 2, 3, \ldots\ . \tag{4}$$

and each $\underline{\nu}$ (for fixed m) goes to infinity with R.

Now it is clear that the numbers k as defined either by equation (1) or (2), are functions of R and ν (for fixed $\underline{a}$) and we shall study the behavior of these functions as $\underline{R}$ and ν go to infinity, subject to relation (4) above.

3. <u>Determination of k for the Transverse Magnetic Modes</u>

We begin with the TM modes and consider the relation (1) above,

$$F(k,R) \equiv J_\nu(kR)\,N_\nu(kR + ka) - J_\nu(kR + ka)\,N_\nu(kR) = 0. \tag{1}$$

We let $\alpha = ka$, $\beta = kR$ and write this as

$$0 = G(\alpha, \beta) \equiv J_\nu(\beta)\,N_\nu(\beta + \alpha) - J_\nu(\beta + \alpha)\,N_\nu(\beta). \tag{1'}$$

Here we shall, for the time being, regard the order ν of the Bessel functions J_ν and N_ν as <u>fixed</u>, but arbitrary, and expand $N_\nu(\beta + \alpha)$ and $J_\nu(\beta + \alpha)$ in powers of α at the point β. We obtain:

$$G(\alpha, \beta) = J_\nu(\beta)\left\{N_\nu(\beta) + \alpha N'_\nu(\beta) + \frac{\alpha^2}{2} N''_\nu(\beta) + \cdots\right\} - N_\nu(\beta)\left\{J_\nu(\beta) + \alpha J'_\nu(\beta) + \frac{\alpha^2}{2} J''_\nu(\beta) + \cdots\right\}, \tag{5}$$

where a symbol of the form P'(t) means $\left.\dfrac{dP(z)}{dz}\right|_{z=t}$.

The functions $J_\nu(z)$ and $N_\nu(z)$ are analytic except at $z = \infty$ and in the case of the second one, at $z = 0$. The Taylor Series representation is therefore valid for all values of α and β, except that $\alpha \neq -\beta$ and $\beta \neq 0$, $\beta \neq \infty$. For α and β chosen

subject to these restrictions, the series must converge absolutely, and by an allowable rearrangement of its terms $G(\alpha, \beta)$ may be written as

$$G(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} D_{o,n}(\beta) \tag{6}$$

where we have adopted the notation, with respect to any variable z,

$$D_{i,j}(z) = J_\nu^{[i]}(z) N_\nu^{[j]}(z) - J_\nu^{[j]}(z) N_\nu^{[i]}(z) , \tag{7}$$

the superscript $[i]$ meaning the ith derivative with respect to z. We shall investigate the expressions in (7) with a view to ultimately finding those terms which may justifiably be regarded as the <u>leading</u> terms in the quite complicated expansion (6).

3.1 Let $u(z)$ and $v(z)$ denote any two independent solutions of the Bessel equation of order ν. Then

$$u'' = -z^{-1} u' + z^{-2} (\nu^2 - z^2) u , \quad \text{and}$$

$$v'' = -z^{-1} v' + z^{-2} (\nu^2 - z^2) v .$$

Substituting these equations into

$$D_{2,r} = u'' v^{[r]} - u^{[r]} v''$$

it follows immediately that

$$D_{2,r} = -z^{-1} D_{1,r} + z^{-2} (\nu^2 - z^2) D_{o,r} .$$

If we set:

$$\rho = z^{-1} , \quad \gamma = z^{-2}(\nu^2 - z^2) = \nu^2 \rho^2 - 1, \tag{8}$$

we may write

$$D_{2,r} = -\rho D_{1,r} + \gamma D_{o,r} , \tag{9}$$

where the D's, ρ, and γ are functions of z.

It is immediate, from equation (7), that

$$D_{i,j} = -D_{j,i} \text{ and } D_{ii} = 0.$$

Furthermore, if we let the symbol δ stand for differentiation with respect to z then:

$$\delta D_{i,j} = D_{i+1,j} + D_{i,j+1} \quad . \tag{10}$$

Now, in particular,

i) $\delta D_{o,n-2} = D_{1,n-2} + D_{o,n-1}$

ii) $\delta D_{o,n-1} = D_{1,n-1} + D_{o,n}$

iii) $\delta D_{1,n-2} = D_{2,n-2} + D_{1,n-1}$

also, from relation (10)

iv) $D_{2,n-2} = -\rho D_{1,n-2} + \gamma D_{o,n-2}$

and finally,

v) $D_{o,n} = (2\delta + \rho) D_{o,n-1} - (\delta^2 + \rho\delta - \gamma) D_{o,n-2}$.

The details in v) are as follows:

$$\begin{aligned}
D_{o,n} &= \delta D_{o,n-1} - D_{1,n-1} && \text{by ii)} \\
&= \delta D_{o,n-1} - \delta D_{1,n-2} + D_{2,n-2} && \text{by iii)} \\
&= \delta D_{o,n-1} - \delta(\delta D_{o,n-2} - D_{o,n-1}) + D_{2,n-2} && \text{by i)} \\
&= \delta D_{o,n-1} - \delta^2 D_{o,n-2} + \delta D_{o,n-1} - \rho D_{1,n-2} + \gamma D_{o,n-2} && \text{by iv)} \\
&= \delta D_{o,n-1} - \delta^2 D_{o,n-2} + \delta D_{o,n-1} - \rho\delta D_{o,n-2} + \rho D_{o,n-1} \\
&\quad + \gamma D_{o,n-2} && \text{by i)}
\end{aligned}$$

We introduce the notation

$$\Delta_n = \frac{\pi}{2} D_{o,n} .$$

The factor $\frac{\pi}{2}$ is chosen to cancel a $\frac{2}{\pi}$ which appears in every $D_{o,n}$. Also we introduce the operators:

$$\sigma = 2\delta + \rho$$

$$\tau = \gamma - \rho\delta - \delta^2 .$$

Then relation v), above, becomes the basic recursion formula:

$$\Delta_n = \sigma \Delta_{n-1} + \tau \Delta_{n-2} \tag{11}$$

3.2. The formulas for Δ_n for the first few integers n are well known, and easily derived. The first two are

$$\Delta_o = 0 \text{ and } \Delta_1 = \rho ,$$

this being the Wronskian of the Bessel equation, multiplied by $\frac{\pi}{2}$. The first five functions $D_{o,n} = \frac{2}{\pi} \Delta_n$ are to be found in Watson, G.N.: Introduction to Bessel Functions, p. 76. From the first two, and the recursion relation (11), we see that $\Delta_2 = \sigma \rho = (2\delta + \rho)\rho = 2\frac{d}{dz}(\frac{1}{z}) + \rho^2 = -\rho^2$,

since ρ is z^{-1}. In general, the expression Δ_n is a polynomial in ρ and ν of increasing order in ρ and ν . It is also a polynomial in, ρ and $\gamma = (\nu^2\rho^2 - 1)$. The form of these polynomials becomes exceedingly complicated with increasing n.

We are interested in the dependence of $G(\alpha, \beta)$ [see equation (1') of article 3.] , on R as R -> ∞ and $\nu = \frac{m\pi R}{b}$. From this point of view, the combination $\nu\rho$ will be found, in the application, to be of zero order in $\frac{1}{R}$, for we intend to apply the theory of article 3.1 to $G(\alpha, \beta)$ so that z will have the value kR. The variable ρ itself will be of first order in $\frac{1}{R}$. Guided by these facts we shall seek out in each Δ_n the terms of total orders one and two in $\frac{1}{R}$, discarding those of higher order.

We shall prove, by induction, that for all $n \geq 1$

$$\text{a)} \quad \Delta_{2n-1} \sim \rho\gamma^{n-1} \tag{12}$$

$$\text{b)} \quad \Delta_{2n} \sim -n(2n-1)\rho^2\gamma^{n-1} - 2n(n-1)\rho^2\gamma^{n-2} ,$$

where the explicit terms on the right are exact, and where all terms omitted are of the form $P\rho^{s+2}\gamma^m$, $s > 0$, P a function of n. There are a finite number of such terms (for each n) and each of them, as remarked, is of at least third order in the ultimate variable $\frac{1}{R}$.

It is convenient to show, first, that for every $m \geq 0$,

$$\text{i)}\quad \sigma\left\{\rho\gamma^m\right\} = -(1+4m)\rho^2\gamma^m - 4m\rho^2\gamma^{m-1}$$

$$\text{ii)}\quad \tau\left\{\rho\gamma^m\right\} \sim \rho\gamma^{m+1} \tag{13}$$

$$\text{iii)}\quad \sigma\left\{\rho^2\gamma^m\right\} \sim 0$$

$$\text{iv)}\quad \tau\left\{\rho^2\gamma^m\right\} \sim \rho^2\gamma^{m+1}$$

$$\text{v)}\quad \sigma\left\{\rho^{s+2}\gamma^m\right\} \sim 0 \text{ and } \tau\left\{\rho^{s+2}\gamma^m\right\} \sim 0, \quad s > 0.$$

the symbol $\sim$ being understood as directly above.

We begin with the simpler operator δ. First,

$$\text{a)}\ \delta\{\gamma\} = -2\nu^2\rho^3 = -2\nu^2\rho^3 + 2\rho - 2\rho = -2\rho(\gamma+1)$$

Next, b) $\delta\{\rho\gamma^m\} = -\rho^2\gamma^m + \rho m\gamma^{m-1}(-2\rho\gamma - 2\rho)$

$$= -(1+2m)\rho^2\gamma^m - 2m\rho^2\gamma^{m-1}\ .$$

$$\text{c)}\ \delta\{\rho^2\gamma^m\} = \rho\delta\{\rho\gamma^m\} + \rho\gamma^m\delta\{\rho\} \sim 0.$$

$$\text{d)}\ \delta^2\{\rho\gamma^m\} \equiv \delta\{\delta\{\rho\gamma^m\}\} \sim 0.$$

Similarly, e) $\delta^2\{\rho^2\gamma^m\} \sim \rho\delta\{\rho\gamma^m\} \sim \rho\delta\{\rho^2\gamma^m\} \sim 0.$

Clearly, f) $\sigma\{\rho^2\gamma^m\} = (\rho + 2\delta)\{\rho^2\gamma^m\} \sim 0$, and

$$\tau\{\rho^2\gamma^m\} = (\gamma - \rho\delta - \delta^2)\{\rho^2\gamma^m\} \sim \rho^2\gamma^{m+1}\ .$$

Finally, $\sigma\{\rho\gamma^m\} = (\rho + 2\delta)\{\rho\gamma^m\} = \rho^2\gamma^m + 2\delta\{\rho\gamma^m\}$

$$= -(1+4m)\rho^2\gamma^m - 4m\rho^2\gamma^{m-1} \quad \text{by b)}$$

and $T\{\rho\gamma^m\} = (\gamma - \rho\delta - \delta^2)\{\rho\gamma^m\} =$

$$\rho\gamma^{m+1} - \rho\delta\{\rho\gamma^m\} - \delta^2\{\rho\gamma^m\} \sim \rho\gamma^{m+1} \quad \text{by d) and e).}$$

Formula v) of 13) is obvious, and we have now completed the proof of relations (13).

Turning now to formulas (12), we see by the first paragraph of this article that they are true in the special case n=1, and we must verify that if they hold for a given n then they hold, also, for n+1.

By the recursion formula (11),

$$\Delta_{2n+1} = \sigma \Delta_{2n} + \tau \Delta_{2n-1} .$$

Assuming that (12) holds for n, we obtain

$$\Delta_{2n+1} = -n(2n-1) \ \sigma \{\rho^2 \gamma^{n-1}\} - 2n(n-1)\sigma \{\rho^2 \gamma^{n-2}\} + \tau \{\rho \gamma^{n-1}\}.$$

Therefore, by formulas (13)

$$\Delta_{2n+1} \sim \rho \gamma^{n}.$$

Notice that in the case n = 1, where the middle term involving γ^{n-2} might seem disquieting, this term is not actually present because of the factor (n-1) in its coefficient. This proves 12 a) for Δ_{2n+1} .

To prove 12 b) for Δ_{2n+2}, we write (11) in the form,

$$\Delta_{2n+2} = \sigma \Delta_{2n+1} + \tau \Delta_{2n} .$$

Now we substitute for Δ_{2n+1} from the preceding paragraph and for Δ_{2n} from 12 b). Then,

$$\Delta_{2n+2} \sim \sigma \{ \rho \gamma^n \} + \tau \{ -n(2n-1)\rho^2 \gamma^{n-1} - 2n \ (n-1)\rho^2 \gamma^{n-2} \} ,$$

Finally by formulas (13),

$$\begin{aligned}
\Delta_{2n+2} &\sim - (1+4n)\rho^2 \gamma^n - 4n \rho^2 \gamma^{n-1} - n(2n-1)\rho^2 \gamma^{n} - 2n(n-1)\rho^2 \gamma^{n-1} \\
&\sim (-1-4n-2n^2 + n)\rho^2 \gamma^n + (-4n-2n^2 + 2n)\rho^2 \gamma^{n-1} \\
&\sim - (n+1)(2n+1)\rho^2 \gamma^n - 2(n+1) \ n \rho^2 \gamma^{n-1} .
\end{aligned}$$

This verifies 12 b) for the case n+1, and concludes the demonstration of (12).

3.3. We shall now substitute these approximations to Δ_n, for every n, in the series $\sum \frac{\alpha^n}{n!} \Delta_n = \sum \frac{\alpha^n}{n!} \frac{\pi}{2} D_{o,n}$ associated with the function $G(\alpha, \beta)$ of equation (6). We shall split this into even and odd terms, depending on n, in accordance with formulas (12). The use of approximations (12) in the terms of the series 6) amounts to a reordering of the terms of an infinite series of the form:

$$(a_{11} + a_{12} + \cdots a_{1n_1}) + (a_{21} + \cdots + a_{2n_2}) + (a_{31} + \cdots + a_{3n_3}) + \cdots$$

into a series $a_{11} + a_{21} + a_{31} + \ldots +$ Remainder; this step calls for careful scrutiny. We postpone the justification*, and turn at once to the series.

We leave to the appendix the question of the legitimacy of the reordering of the series of terms in $\sum \frac{\alpha^n}{n!} \Delta_n$ which we are about to perform. First we separate $\sum_{n=1} \frac{\alpha^n}{n!} \Delta_n$ into the sum of an "odd" and "even" series,

$\sum_{m=1}^{\infty} \frac{\alpha^{2m-1}}{(2m-1)!} \Delta_{2m-1}$ and $\sum_{m=1}^{\infty} \frac{\alpha^{2m}}{(2m)!} \Delta_{2m}$. Then we substitute for each of the Δ's, the values from 12.

Hence,

$$\sum^{\infty} \frac{\alpha^n}{n!} \Delta_n = \rho \sum_{m=1}^{\infty} \frac{\alpha^{2m-1}}{(2m-1)!} \gamma^{m-1} - \rho^2 \sum_{m=1}^{\infty} \frac{\alpha^{2m}}{(2m)!} \left(m(2m-1)\gamma^{m-1} + 2m(m-1)\gamma^{m-2} \right)$$

+ a remainder; each term of this remainder is of the form $\rho^s f_s(\alpha, \gamma)$, $s \geq 3$, with f_s an appropriate series in α whose coefficients are polynomials in γ.

We introduce $x^2 = -\alpha^2 \gamma$; as we shall see later x is real and positive, in applications. By simple manipulations using, in the last series, the relation $n-1 = \frac{1}{2}(2n-1) - \frac{1}{2}$, we may express this as

$$\sum_{n=1} \frac{\alpha^n}{n!} \Delta_n = \frac{\alpha\rho}{x} \sum_{n=1} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} - \frac{\alpha^2\rho^2}{2} \sum_{n=1} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!} + \frac{1}{2} \frac{\alpha^2\rho^2}{\gamma x} \sum_{n=1} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$- \frac{\alpha^2\rho^2}{2\gamma} \sum_{n=1} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!} + \text{remainder.}$$

Substituting the trigonometric functions appropriate to the summations,

$$\sum \frac{\alpha^n}{n!} \Delta_n = \alpha\rho \frac{\sin x}{x} + \frac{\alpha^2\rho^2}{2} \left(\frac{\sin x}{\gamma x} - (1 + \frac{1}{\gamma}) \cos x \right) + \text{remainder.}$$

or

$$\sum \frac{\alpha^n}{n!} \Delta_n \sim \alpha\rho \frac{\sin x}{x} + \frac{\alpha^2\rho^2}{2} \left[\frac{\sin x}{x} - (1 + \frac{1}{\gamma} \cos x) \right] \tag{15}$$

---

* The justification is given in the appendix.

3.4. We shall substitute the result (15) into our fundamental equation (1), or its equivalent (1') of article 3, first multiplying through by $\frac{\pi}{2}$ to eliminate this factor, as was done in article 3.1. We shall need to recall that $\alpha = ka$, and that $\beta = kR$ is now the value of $z = \rho^{-1}$ (see (8) of article 3).

This gives us

$$0 \sim \frac{a}{R}\ \frac{\sin x}{x} + \frac{1}{2}\ \frac{a^2}{R^2}\left\{\frac{\sin x}{\gamma x} - \left(1 + \frac{1}{\gamma}\right)\cos x\right\} \quad , \qquad (16)$$

where the symbol $\sim$ indicates that terms, each of them of the form $R^{-s}\varphi(k), s > 2$, have been omitted. Observe, in this connection, that

i) $\gamma = \nu^2 \rho^2 - 1 = \frac{c^2}{k^2} - 1$,

where $c = \frac{\pi m}{b}$, by formula (4) of article 2.1;

and ii) $x = \sqrt{-\alpha^2\gamma} = a\sqrt{k^2 - c^2}$, so that γ and x are formally independent of R.

It is clear that the right-hand side of formula (16) has precisely the character of a Taylor's expansion of the function F(k,R), of article 3, in powers of $\frac{1}{R}$ about $\frac{1}{R} = 0$, with k constant but with ν, the order of the Bessel functions depending on R(through $\nu = cR$, as above). The relation (16) defines k as a many-valued function of R.

We observe as a special case of relation (16) that if we multiply through by R and then let R become infinite we have

$$a\ \frac{\sin x}{x} = 0$$

or

$$\sin x = 0$$

Since $x = \sqrt{-\alpha^2\gamma}$ we must have

$$\sqrt{-\alpha^2\gamma} = n\pi$$

or $$a\sqrt{k^2 - c^2} = n\pi$$

Since

$$c = \frac{m\pi}{b},$$

$$k = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}$$

Which is the correct formula for the cut off frequency of a rectangular guide. This is precisely the result[12] we should have when R becomes infinite and therefore confirms the fact that equation (16) is correct at least to the first order term.

It is more important to observe that equation (16) gives us the value of k for coaxial sectors to second order terms so that we now have information about coaxial sectors and their relationship to the limiting rectangular guide.

4. Explicit Determination of k

To evaluate the effect on k of deformation of the wave guide we seek an explicit expression for k from relation (16). We observe first that the variation of k with R, the inner radius of the coaxial sector, would necessarily be slight; for R must change considerably and ultimately become infinite to produce a physical change from slight deformation to a perfect rectangle. It is more significant to consider the variation of k with $\frac{1}{R}$ which is the curvature of the inner cylinder. Let $\frac{1}{R}$ then be a new variable C and we write equation (16) as

$$0 = a\,\frac{\sin x}{x}\,C + \frac{1}{2}\,a^2\left\{\frac{\sin x}{x} - \left(1 + \frac{1}{\gamma}\right)\cos x\right\}C^2 \qquad (16')$$

to second order in C.

It would be practically quite difficult to solve for k directly. Instead we shall seek a Taylor's expansion for k in terms of C in the neighborhood of C = 0. We may divide by C. Then, since by Taylor's theorem,

$$k(C) = k(0) + \left.\frac{dk}{dC}\right|_{C=0} C + \ldots\ldots$$

we seek k(0) and $\left.\frac{dk}{dC}\right|_{c=0}$; k(0) was obtained in article 3.4.

It will be noted that there is really a double infinity of values for k(0), one for each mode. Likewise there will be a double infinity of values of k(C), each one of these approaching one and only one value of k(0) as C approaches 0.

To find $\left.\frac{dk}{dC}\right|_{C=0}$ we use equation (16') after division by C and differentiate implicity using the facts that $x = a\sqrt{k^2-c^2}$ and $\gamma = \frac{c^2}{k^2} - 1$, wherein a and c are fixed.

This gives

$$\left.\frac{dx}{dC}\right|_{C=0} = -\frac{a}{2}\left\{\frac{x\sin x - \left(x^2 + \frac{x^2}{\gamma}\right)\cos x}{x\cos x - \sin x}\right\}_{C=0}$$

12. Ramo and Whinnery, loc.cit., p. 342, formula [4] .

Since $x^2 = a^2k^2 - a^2c^2$, $\frac{dx}{dC} = \frac{a^2k}{x}\frac{dk}{dC}$. Hence

$$\left.\frac{dk}{dC}\right|_{C=0} = -\frac{x}{2ak}\left.\left\{\frac{x \sin x - \left(x^2 + \frac{x^2}{\gamma}\right)\cos x}{x \cos x - \sin x}\right\}\right|_{C=0}$$

At C = 0 or R = ∞, $x = m\pi$ (see art. 3.4). Also $\gamma = \frac{c^2}{k^2} - 1$ with $c = \frac{m\pi}{b}$.

Hence $\left.\frac{dk}{dC}\right|_{C=0} = \frac{x^2}{2ak(0)}\left(1 + \frac{1}{\gamma}\right)$

Since $x^2 = -\alpha^2\gamma$ and $\alpha = ka$

$$\left.\frac{dk}{dC}\right|_{C=0} = -\frac{ac^2}{2k(0)}$$

Finally

$$k(C) = k(0) - \frac{ac^2}{2k(0)}\,C + \dots \qquad (17)$$

where $k(0) = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}$, $c = \frac{m\pi}{b}$,

and $C = \frac{1}{R}$.

It should be remarked first that the accuracy to which k is obtained in formula (17) is of the same order as the accuracy of equation (16). Were we to seek the next term in the Taylor's expansion for k(C) we would however have to use the term in $\frac{1}{R^3}$ not present in equation (16). The coefficient of $\frac{1}{R^3}$ in the expansion (16) was actually calculated and found to be exceedingly cumbersome.

It is now possible, by means of formula (17), to give some estimate of the effect of curvature on the value of k. The variation in k(C) is given by $\Delta k(C) = \frac{ac^2}{2k(0)}$. Now $k(0) = \sqrt{\frac{n^2\pi^2}{a^2} + c^2} \geq c$. Hence $\frac{\Delta k(C)}{k(0)} \leq \frac{aC}{2}$,

i.e. the ratio of the variation in k(C) to k(0) does not exceed $\frac{aC}{2}$. Thus, for example, for a coaxial sector with $R \geq 5a$, this gives a less than ten percent variation from a rectangular guide. The same observations apply to the cut-off wave-length in air, $\lambda = \frac{2\pi}{k}$. These modes thus possess a reasonable amount of stability in cut-off wave length under the deformations considered in this paper.

5. The Transverse Electric Modes for the Coaxial Sector

The preceding discussion has been devoted to the transverse magnetic modes. The transverse electric modes for the coaxial sector are determined by solutions for k of the equation

$$\varphi(k,R) = \Gamma'(\alpha,\beta) = J_\nu'(\beta)\,N_\nu'(\beta+\alpha) - J_\nu'(\beta+\alpha)\,N_\nu'(\beta) = 0 \tag{2'}$$

where again $\alpha = ka$ and $\beta = kR$. As in the treatment of the transverse magnetic modes we shall let

$$D_{i,j}(z) = J_\nu^i(z)\,N_\nu^j(z) - J_\nu^j(z)\,N_\nu^i(z)$$

where again i and j denote differentiation with respect to the argument, and for convenience here let us use the notation

$$\Delta_{i,j}(z) = \frac{\pi}{2}\,D_{i,j}(z)\ .$$

If now, as in the transverse magnetic case we apply Taylor's theorem to $N_\nu'(\beta+\alpha)$ and $J_\nu'(\beta+\alpha)$ and rearrange terms as before we obtain

$$\Gamma(\alpha,\beta) = \frac{2}{\pi}\sum_{1}^{\infty}\frac{\alpha^n}{n!}\,\Delta_{1,n+1}(\beta) \tag{18}$$

The problem now, as before, is to obtain expressions for the first and second order terms in $\frac{1}{R}$ which are involved in each of the Δ's .

By relation (10),

$$D_{i+1,j} = \delta D_{i,j} - D_{i,j+1}$$

Hence

$$\Delta_{1,2n} = \delta\Delta_{0,2n} - \Delta_{0,2n+1}$$

By equation (12)

$$\Delta_{1,2n} \sim -\rho\,\delta^{-n}$$

since $\delta\left(=\frac{d}{dz}\right)$ operating on $\Delta_{0,2n}$ produces no first order terms in ρ. (see formulas (12) and (13 v.)) .

Likewise

$$\Delta_{1,2n-1} = \delta\,\Delta_{0,2n-1} - \Delta_{0,2n}$$

Using equation (13v) we get

$$\Delta_{1,2n-1} \sim -\rho^2 \gamma^{n-1} + (n-1)\,\rho\,\gamma^{n-2}(-2\rho)(\gamma+1) + n(2n-1)\rho^2\gamma^{n-1}$$
$$+2n(n-1)\rho^2\gamma^{n-2}$$

$$\Delta_{1,2n-1} \sim (2n^2-3n+1)\rho^2\gamma^{n-1} + 2(n^2-2n+1)\rho^2\gamma^{n-2} \tag{19b}$$

We shall now use the results (19a,19b) in a manner completely analogous to the use of (12a,12b) in the discussion of the transverse magnetic mode, rearranging terms as was done there. We obtain,

$$\frac{\pi}{2}\Gamma(\alpha,\beta) \sim -\rho \sum_{1}^{\infty} \frac{\alpha^{2m-1}}{(2m-1)!}\gamma^m + \rho^2 \sum_{2}^{\infty} \frac{\alpha^{2m-2}}{(2m-2)!}$$

$$\left[(2m^2-3m+1)\gamma^{m-1} + 2(m^2-2m+1)\gamma^{m-2}\right]$$

$$\sim -\rho\alpha\gamma\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}}{(2m-1)!} + \rho^2 \sum_{1}^{\infty}\frac{\alpha^{2m}}{2m!}\left[(2m^2+m)\gamma^m + 2m^2\gamma^{m-1}\right]$$

$$\sim -\rho\alpha\gamma\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}}{(2m-1)!} + \rho^2\sum_{1}^{\infty}\frac{\alpha^{2m}\gamma^{m-1}}{2m!}\left[(2m^2+m)\gamma + 2m^2\right]$$

$$\sim -\rho\alpha\gamma\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}}{(2m-1)!} + \rho^2\alpha^2\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}}{2m!}\left[(\gamma+1)2m^2 + m\gamma\right]$$

$$\sim -\rho\alpha\gamma\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}}{(2m-1)!} + \rho^2\alpha^2\,2(\gamma+1)\sum_{1}^{\infty}\frac{(\alpha^2\gamma)^{m-1}m^2}{2m!} + \rho^2\alpha^2\gamma\sum_{1}^{\infty}\frac{m}{2m!}(\alpha^2\gamma)^{m-1}$$

Expressions such as these have been encountered above in article 3.3. Again we let $x^2 = -\alpha^2\gamma$ and make a few minor algebraic changes. Then,

$$\frac{\pi}{2}\Gamma(\alpha,\beta) \sim -\rho\alpha\gamma\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}}{(2m-1)!} + \rho^2\alpha^2\,2(\gamma+1)\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}m^2}{2m!} +$$
$$+\rho^2\alpha^2\gamma\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}m}{2m!}$$

$$\frac{\pi}{2}\Gamma(\alpha,\beta)\sim-\rho\alpha\gamma\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}}{(2m-1)!}+\frac{\rho^2\alpha^2 2(\gamma+1)}{4}\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}2m}{(2m-1)\,!}$$

$$+\frac{\rho^2\alpha^2\gamma}{2}\sum_{1}^{\infty}\frac{(-1)^{m-1}x^{2m-2}}{(2m-1)\,!} \qquad (20)$$

By replacing 2n in the middle term by (2m-1), breaking up the summation into two terms, and using the series expressions for sin x and cos x, this expression can be rewritten as

$$\frac{\pi}{2}\Gamma(\alpha,\beta)\sim-\rho\alpha\gamma\frac{\sin x}{x}+\frac{\rho^2\alpha^2(\gamma+1)}{2}\left[\frac{\sin x}{x}+\cos x\right]+\frac{\rho^2\alpha^2\gamma}{2}\frac{\sin x}{x}$$

$$\sim-\rho\alpha\gamma\frac{\sin x}{x}+\frac{\alpha^2\rho^2}{2}\left[(\gamma+1)\left(\frac{\sin x}{x}+\cos x\right)+\gamma\frac{\sin x}{x}\right] \qquad (21)$$

By letting $\rho=\frac{1}{z}$ take on the value $\frac{1}{kR}=\frac{1}{\beta}$ we obtain $\Gamma(\alpha,\beta)$ exhibited to second order terms in $\frac{1}{R}$ just as $G(\alpha,\beta)$ was previously exhibited. Expression (21) set equal to 0 defines k as a function of $\frac{1}{R}$. To obtain a Taylor's expansion for the wave number k in terms of the curvature $C=\frac{1}{R}$, as was done in the case of the transverse magnetic modes, we first substitute ka for α, $\frac{1}{kR}$ for ρ, divide through by $\frac{a}{R}$, and, finally, replace $\frac{1}{R}$ by C in expression (21), and obtain

$$0=\gamma\frac{\sin x}{x}+\frac{aC}{2}\left[(\gamma+1)\left(\frac{\sin x}{x}+\cos x\right)+\gamma\frac{\sin x}{x}\right] \qquad (22)$$

wherein $x^2=-k^2a^2\gamma=-k^2a^2(\frac{c^2}{k^2}-1)$ and $c=\frac{m\pi}{b}$. To obtain k(0) set C = 0 giving

$$\gamma\frac{\sin x}{x}=0$$

Now $\gamma=0$ implies $x=0$ and $\sin x=0$ implies $x=n\pi$, $n=0,\pm1,\pm2,\ldots$.

It follows that,

$$k(0)=\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}}\,.$$

This expression is precisely the one which should hold for the limiting case of the rectangle.

To obtain $\left.\frac{dk}{dC}\right|_{C=0}$ we differentiate equation (22) with respect to C, remembering that x is a function of C and set C = 0. To change from $\frac{dx}{dC}$ to $\frac{dk}{dC}$ we

need multiply by $\frac{x}{a^2 k}$. We obtain,

$$\frac{dx}{dC} = -\frac{a}{2}\frac{\gamma + 1}{\gamma} x$$

and

$$\left.\frac{dk}{dC}\right|_{C=0} = \frac{ac^2}{2k(0)} .$$

This gives for k:

$$k(C) = k(0) + \frac{ac^2}{2k(0)} + \cdots \tag{23}$$

The remarks made in article 4 on the magnitude of the variation in k apply here too.

In addition there are one or two interesting observations which can be made for the very practical case of the $TE_{0,1}$ and $TE_{1,0}$ modes. Each of these is included in the above theory, the one by letting m = 0 and the other by letting n = 0. It is clear from the very origin of m that the value of m determines the variations of the field in the direction of varying φ [see equation (4)]. Hence the mode m = 0, n = 1 corresponds roughly to Fig. 3 and should in the limit give rise to the rectangular guide mode $TE_{0,1}$ which has its electric field horizontal. The cut-off frequency of this mode ought to be insensitive to the curvature of the sides; for in the true rectangle this dimension does not affect the cut-off frequency. Formula (23) is in accord with this reasoning because the value m = 0 implies C = 0 and for this value of C the variation in cut-off frequency is 0 (to the first order) regardless of the curvature C.

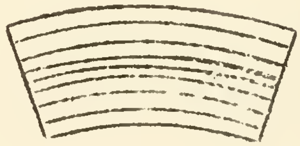

FIG. 3

One would not expect insensitivity to curvature in the $TE_{1,0}$ mode, namely, the one in which the electric lines are perpendicular to the curved sides (Fig. 4) for this dimension does affect cut-off frequency. By formula (23) the case n = 0 does show a variation in cut-off frequency with curvature.

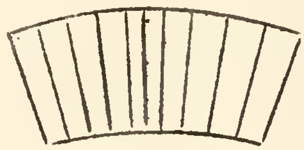

FIG. 4

It is very interesting that for $m = 0$ all of the above theory holds. It implies $\nu = 0$ and $c = 0$ throughout. Returning to equation (2) it means that we have obtained a Taylor's expansion for

$$J_0'(kR)\, N_0'\,(ka + kR) - J_0'\,(kR + ka)\, N_0'\,(kR) = 0$$

around $R = \infty$. This particular expansion can be checked against the usual asymptotic expansions of J_n and N_n as given by Watson[13], for example, for in this special case the order of the functions does not vary with the argument. This check has been made and confirms the correctness of relation (22).

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13. Watson, G. N.: Theory of Bessel Functions, Macmillan, N.Y., 1945 p.199.

Appendix

This appendix will justify the rearrangement of terms in the function $\sum_n \frac{\alpha^n}{n!} \Delta_n(\nu, \rho)$. This rearrangement was utilized in the body of the paper to convert the above expression which is a power series in α, with coefficients dependent upon ν and ρ, into a power series in ρ with coefficients dependent on k and c. As the body of the paper shows, when ν and ρ are replaced by their values in terms of R, the combination $\nu\rho$ is independent of R, the latter being the essential variable for the purposes of this paper.

Actual calculation of the quantities $\Delta_2, \Delta_3, \Delta_4, \ldots$ by operating with the basic recursion formula (11) of the paper, namely,

$$\Delta_n = \sigma \Delta_{n-1} + \tau \Delta_{n-2}$$

wherein $\sigma = 2\delta + \rho$ and $\tau = \nu^2 \rho^2 - 1 - \rho\delta - \delta^2$, δ being $\frac{d}{dz}$ and ρ being $\frac{1}{z}$, indicates that Δ_n for even n can be expressed as a polynomial in even powers of ρ with coefficients which are polynomials in even powers of ν. Likewise for odd n, Δ_n can be expressed as a polynomial in odd powers of ρ with coefficients which are polynomials in even powers of ν. More formally stated, we have

Theorem A

$$\Delta_{2n-1}(\rho) = \sum_{i=0}^{n-1} P_i^{2n-1} \rho^{2n-2i-1}$$

wherein

$$P_i^{2n-1} = \sum_{j=0}^{n-1-i} a_{i,j}^{2n-1} \nu^{2(n-1-i-j)}$$

and

$$\Delta_{2n}(\rho) = \sum_{i=0}^{n-1} P_i^{2n} \rho^{2n-2i}$$

wherein

$$P_i^{2n} = \sum_{j=0}^{n-1-i} a_{i,j}^{2n} \nu^{2(n-1-i-j)}$$

Moreover, the coefficients $a_{i,j}^n$ satisfy the relations

a) $a_{i,j}^{2n+1} = -(4n-4i-1)\, a_{i,j-1}^{2n} + a_{i,j}^{2n-1} - (2n-2i-1)^2\, a_{i,j-1}^{2n-1} - a_{i-1,j}^{2n-1}$

b) $a_{i,j}^{2n+2} = -(4n-4i+1)\, a_{i,j}^{2n+1} + a_{i,j}^{2n} - (2n-2i-2)^2\, a_{i,j-1}^{2n} - a_{i-1,j}^{2n}$,

with the understanding that $a^m_{i,j} = 0$ when $i + j > \frac{m-1}{2}$ or $i < 0$, or $j < 0$, or $m < 1$.

Proof. The proof of this theorem is by mathematical induction. We have the recursion formula (11) which relates Δ_n to Δ_{n-1} and Δ_{n-2}. The form of Δ_n is known for the first few Δ_n's and agrees with the statement of the theorem. We now assume that the form holds for Δ_{2n-1} and Δ_{2n} and apply the basic recursion formula to show that it holds for Δ_{2n+1} and Δ_{2n+2}.

One notes that ν is independent of z and hence the differentiation with respect to z which is called for in the use of the operators σ and τ affects only powers of $\rho = \frac{1}{z}$. If we calculate the expansion of Δ_{2n+1} by the use of formula (11) and collect in powers of ρ we find for the coefficient of $\rho^{2n-2i+1}$ the relation,

$$P_i^{2n+1} = -(4n-4i-1)\, P_i^{2n} + (\nu^2 - \overline{2n-2i-1}^{\,2})\, P_i^{2n-1} - P_{i-1}^{2n-1} .$$

Similarly, we obtain, for the coefficient of $\rho^{2n-2i+2}$ in Δ_{2n+2}, the relation

$$P_i^{2n+2} = -(4n-4i+1)\, P_i^{2n+1} + (\nu^2 - \overline{2n-2i}^{\,2})\, P_i^{2n} - P_{i-1}^{2n} .$$

We now substitute in the right hand side of these relations the appropriate polynomials in ν, and collect terms. This exhibits each left hand side as a simple polynomial in ν. It is easy to verify that the coefficients of these polynomials satisfy Theorem A.

We shall now need to investigate the pattern of signs of the coefficients of the polynomials P_i^m with a view to showing that we are dealing with an absolutely convergent series. This will be crucial in justifying our rearrangement of terms of this series. We next prove

Theorem B

(1) the sign of $a^m_{i,j} = (-1)^{m+i+1}$, $m = 1,2,3$
$0 \leqq i \leqq \frac{m-1}{2}$

(2) the sign of $\left\{ a^{2n-1}_{i,j-1} + (2n-2i-2)\, a^{2n-2}_{i,j-2} \right\} = (-1)^i$, $i + j \leqq n-1$

(3) the sign of $\left\{ a^{2n}_{i,j-1} + (2n-2i-1)\, a^{2n-1}_{i,j-1} \right\} = (-1)^{i+1}$, $i + j \leqq n-1$

Proof. It is (1) only, of theorem B, which will be needed ultimately; however the nature of the proof forces us to be concerned with the parts (2) and (3). The proof is by mathematical induction. By inspection of $\triangle_{2n-1}$ and $\triangle_{2n}$ when n = 1 we find all three statements to be true. We now assume these statements true for 2n, 2n-1 and 2n-2 and proceed to the main step of the inductive argument.

It follows from relation a) of Theorem A, with j replaced by j-1 and the terms regrouped, that for all i,j

$$a^{2n+1}_{i,j-1} + (2n-2i)\, a^{2n}_{i,j-2} = -(2n-2i-1)\left\{ a^{2n}_{i,j-2} + (2n-2i-1)\, a^{2n-1}_{i,j-2} \right\} + a^{2n-1}_{i,j-1} - a^{2n-1}_{i-1,j-1} \quad . \qquad \text{a)}$$

By the induction assumption (3) the sign of the first term on the right is $(-1)^{i+2} = (-1)^i$; the sign of the second term, by assumption (1) is $(-1)^{2n-1+i+1} = (-1)^i$; and the sign of the third term is $(-1)^{1+(2n-1)+(i-1)+1} = (-1)^i$. It follows that assumption (2) holds for n+1.

By the induction assumption (1) the sign of $a^{2n}_{i,j-2}$ is $(-1)^{i+1}$ and the sign of the entire left side of a) is $(-1)^i$ by the previous paragraph. It follows therefore that the term $a^{2n+1}_{i,j-1}$ on the left side of a) has the sign $(-1)^i$. This step completes the inductive proof of statement (1) for the case where m = 2n+1.

To establish statement (3) of the theorem we use relation (b) of Theorem A. If in that relation we replace j by j-1 and rearrange terms slightly we obtain

$$a^{2n+2}_{i,j-1} + (2n-2i+1)\, a^{2n+1}_{i,j-1} =$$

$$-(2n-2i)\, a^{2n+1}_{i,j-1} - (2n-2i-2)\, a^{2n}_{i,j-2} + a^{2n}_{i,j-1} - a^{2n}_{i-1,j-1}$$

$$= -(2n-2i)\left\{ a^{2n+1}_{i,j-1} + (2n-2i)\, a^{2n}_{i,j-2} \right\} + 2(4n-4i-2)\, a^{2n}_{i,j-2} + a^{2n}_{i,j-1} - a^{2n}_{i-1,j-1}$$

Since we have already established relation (2) for n+1, it follows that the sign of the first term on the right is $(-1)^{i+1}$. This is also the sign of each of the remaining terms in view of the inductive assumption for relation (1). Hence the right side and therefore the left side has the sign $(-1)^{i+1}$, which establishes relation (3) for n+1. At the same time, since $a^{2n+1}_{i,j-1}$ has the sign $(-1)^i$ by the inductive proof

of relation (1) when m is odd, it must be that the sign of $a_{i,j-1}^{2n+2}$ is $(-1)^{i+1}$ and this is the proof of relation (1) for $m = 2n+2$, which completes the entire proof of that relation.

Theorem B is now proved in every part.

In consequence of (1) of Theorem B, the coefficients $a_{i,j}^{m}$ in the polynomials P_i^m in Theorem A are all of one sign, depending on m and i, and we obtain,

Corollary I:

$$(-1)^{i} P_i^{2n-1} = \sum_{j=1}^{n-i-1} \left| a_{i,j}^{2n-1} \right| \nu^{2(n-1-i-j)}$$

$$(-1)^{i+1} P_i^{2n} = \sum_{j=1}^{n-i-1} \left| a_{i,j}^{2n} \right| \nu^{2(n-1-i-j)} .$$

We now write $\rho = \sqrt{-1}\ r$. Then substituting the formulas of Corollary I into the polynomials Δ_{2n-1} and Δ_{2n}, in Theorem A, we obtain

Corollary II:

$$\Delta_{2n-1}(\sqrt{-1}\ r) = (-1)^{n-1}\sqrt{-1}\ Q_{2n-1}(r, \nu) ,$$

$$\Delta_{2n}(\sqrt{-1}\ r) = (-1)^{n+1} Q_{2n}(r, \nu) .$$

where $Q_m(r, \nu)$ is a polynomial in r and ν whose coefficients are all positive. It is clear that these coefficients are, in fact, $\left| a_{i,j}^{m} \right|$.

Let us now write $\alpha = \sqrt{-1}\ A$. Then

$$\alpha^{2n-1} \Delta_{2n-1}(\sqrt{-1}\ r) = (\sqrt{-1})^{2n-1}(\sqrt{-1}\ (-1)^{n-1} A^{2n-1} Q_{2n-1}(r, \nu)$$

$$= -A^{2n-1} Q_{2n-1}(r, \nu) \text{ and}$$

$$\alpha^{2n} \Delta_{2n}(\sqrt{-1}\ r) = -A^{2n} Q_{2n}(r, \nu).$$

We formulate these results in the following

Corollary III: for $\alpha = \sqrt{-1}\ A$ and $\rho = \sqrt{-1}\ r$,

$$-\alpha^{m} \Delta_{m}(\rho) = A^{m} Q_{m}(r, \nu), \quad m = 1,2,3,\ldots\ldots ,$$

where the coefficients in the polynomial Q_m are all positive.

We are now near the conclusion. We recall that the Bessel functions $J_\nu(z)$ and $N_\nu(z)$ are analytic functions of the argument z everywhere except at $z = \infty$ and, for the second of these functions at $z = 0$. It follows at once that

$$G(\alpha, \beta) = J_\nu(\beta)\, N_\nu(\alpha+\beta) - J_\nu(\alpha+\beta)\, N_\nu(\beta)$$

is analytic in α and β excepting possibly at loci of the form:

$$\alpha = 0, \alpha = \infty; \ \beta = 0, \beta = \infty; \alpha + \beta = 0, \alpha + \beta = \infty .$$

We already know (6) that

$$\frac{2}{\pi} G(\alpha, \rho^{-1}) = \sum_{m=1}^{\infty} \frac{\alpha^m}{m!} \Delta_m(\nu, \rho) .$$

The series on the right converges for all α and $\rho = \beta^{-1}$ which are not on the exceptional loci.

Then, by Corollary III,

$$- \frac{2}{\pi} G(\sqrt{-1}\, A, \ 1/\sqrt{-1}\, r) = \sum_{m=1}^{\infty} \frac{A^m}{m!} Q_m(\nu, r) .$$

This power series in A converges for every A, for every fixed r, with exceptions as above; provided, of course, that in the summation we regard the terms in each $Q_m(\nu, r)$ as locked in a parenthesis, the entire polynomial serving as coefficient for A^m. But now, since each coefficient of $Q_m(\nu, r)$ is positive, we may regard the series on the right as a simple sequence of terms (A and r being thought of as fixed, arbitrary, real positive numbers) and we may certainly rearrange the terms of this simple sequence in any order, and reintroduce parentheses in any way we wish. The resulting series will converge to the same value as the original series.[14] This completes the justification of the rearrangement of terms to which we subjected the series associated with the TM modes. An entirely analogous argument disposes of the case of the TE modes. We shall not set down the details.

It is worth remarking that we can now recognize as the leading terms of a Taylor's expansion about $R = \infty$, the formula

$$\frac{\pi}{2}\left[J_\nu(ka)\, N_\nu(ka + kR) - J_\nu(ka + kR)\, N_\nu(ka)\right]_{\nu = cR} =$$
$$= \frac{a}{R}\, \frac{\sin x}{x} + \frac{1}{2}\frac{a^2}{R^2}\left\{\frac{\sin x}{\gamma x} - \left(1 + \frac{1}{\gamma}\right)\cos x\right\} + \dots ,$$

where $\gamma + 1 = c^2/k^2$ and $x^2 = a^2(k^2 - c^2)$.

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14. See, for example, Bromwich, T. J.: An Introduction to the Theory of Infinite Series, Macmillan & Co., Ltd., London, 1908, articles 31 and 33.